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NUMERICAL SOLUTION OF THE MULTIDIMENSIONAL CROSS-DIFFUSION PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. This article investigates the asymptotic behavior of self-similar solutions of a nonlinear cross-diffusion system with nonlocal boundary conditions. The main term of the asymptotics of self-similar solutions has been determined. For the numerical study of the problem under consideration, a method for selecting the initial approximation corresponding to the iterative process is proposed. Numerical calculations and analysis of the results were carried out using asymptotic formulas as an initial approximation for the iteration process.

Keywords: Cross-diffusion, nonlocal boundary condition, self-similar.

1 INTRODUCTION

The process of cross-diffusion is a phenomenon in which the components of a mixture interact with each other through interdependent spatial distribution. In cross-diffusion, one component influences the movement of another, unlike normal diffusion, where each component moves independently from high to low concentration. This interaction is especially important in multiphase systems such as chemical solutions, biological cell populations, or ecosystem processes.

Cross-diffusion is a phenomenon in which the components of a mixture interact with each other through interdependent spatial distributions. This interaction is particularly important in multiphase systems such as chemical solutions, biological cell populations, or ecosystem processes. Many areas of the natural sciences, including biology, ecology, chemistry, and physics, utilize cross-diffusion models. These models better describe complex interactions in multicomponent systems, where the existence and behavior of one component affects the mobility of another. A brief description of cross-diffusion models and their applications in several areas of natural science:

In chemistry and reaction physics, cross-diffusion models describe the interactions of molecules whose ions can influence each other through concentration gradients, creating new chemical patterns. The Schnakenberg-Turing model is popular and employs cross-diffusion to explain periodic patterns in reaction-diffusion systems. This model is applicable to reactions where one reactant causes the other to move, forming self-sustaining patterns. It demonstrates how cross-diffusion can generate processes that continuously repeat themselves and patterns related to chemical waves and Belousov-Zhabotinsky reactions.;

In biology and ecology, cross-diffusion models are used to model populations in the presence of other species and how cells migrate in response to chemical signals. For example, the Keller-Segel model of chemotaxis describes how bacteria and other cells move to areas where chemical attractants produced by other cells are present in high concentrations. This model explains the formation and maintenance of aggregation patterns in biological populations, from bacterial colonies to animal tissues. Ecology uses cross-diffusion models to explain the interaction of species in spatial ecosystems. Cross-diffusion models of competition and cooperation can explain spatial patterns such as stripes or patches in plant or animal communities. Cross-diffusion models provide a powerful tool for describing and predicting the behavior of complex systems where components are interdependent.

2 PROBLEM STATEMENT

The article examines the asymptotic behavior of solutions of a multidimensional cross-diffusion system with nonlinear boundary conditions:

$$\frac{\partial u}{\partial t}=\nabla\left(\upsilon^{m_1-1}\nabla u\right),\quad \frac{\partial \upsilon}{\partial t}=\nabla\left(u^{m_2-1}\nabla \upsilon\right),\quad x\in R_+,\ t>0, \tag{1}$$

$$-\upsilon^{m_1-1}\frac{\partial u}{\partial x}(0,t)=u^{q_1}(0,t),\quad -u^{m_2-1}\frac{\partial \upsilon}{\partial x}(0,t)=\upsilon^{q_2}(0,t),\ t>0, \tag{2}$$

$$u(x,0)=u_0(x),\quad \upsilon(x,0)=\upsilon_0(x),\quad x\in R_+, \tag{3}$$

where $R_+^N=\left\{(x_1,x')\,|\,x'\in R^{N-1},\ x_1>0\right\}$ $m_i>1,$ $q_i>0\,(i=1,2),$ u_0 and $\upsilon_0(x)$ are non-negative continuous functions with compact support in R_+^N .

The process of cross-diffusion means that the spatial shift of one object, characterized by one variable, occurs due to the diffusion of another object, characterized by another variable [1].

The laws of cross-diffusion are observed in various fields of natural sciences; for example, in physical systems (plasma physics) [1-3], in chemical systems (dynamics of electrolytic solutions), in biological systems (cross-diffusion transfer, dynamics of population systems), in ecology (dynamics of forest age structure), in seismology - the Burridge-Knopoff model describing the interaction of tectonic plates [4-7]. Cross-diffusion mathematical models are widely used in studying the dynamics of biological populations and the movement of tectonic plates [4, 5].

The study of conditions for global solvability and non-solvability of problem (1) - (3) for various values of numerical parameters has been the subject of many studies [1-16]. In [8, 9], the authors examined the conditions for global solvability and non-solvability of the solution over time and determined the estimate of the solution near the blow-up time of the nonlocal diffusion problem.

$$u_t=u_{xx},\quad \upsilon_t=\upsilon_{xx},\quad x>0,\ 0<T<0, \tag{4}$$

$$-u_x(0,t)=u^{\alpha}\upsilon^{p},\quad -\upsilon_x(0,t)=u^{q}\upsilon^{\beta},\quad 0<t<T, \tag{5}$$

$$u(x,0)=u_0(x),\ \upsilon(x,0)=\upsilon_0(x),\ x>0. \tag{6}$$

If $pq\le(1-\alpha)(1-\beta)$, then it is proven that each solution of problem (4) - (6) is global. The following problems were studied in [10]:

$$u_t=\left(u^n\right)_{xx},\quad \upsilon_t=\left(\upsilon^k\right)_{xx},\quad x\in R_+,\ t>0, \tag{7}$$

$$-\left(u^n\right)_x(0,t)=\upsilon^p(0,t),\quad -\left(\upsilon^k\right)_x(0,t)=u^q(0,t),\quad t>0, \tag{8}$$

$$u(x,0)=u_0(x),\quad \upsilon(x,0)=\upsilon_0(x),\quad x\in R_+. \tag{9}$$

It is shown that the solution to problem (7) - (8) is global in $pq\le(n+1)(k+1)/4$. For the numerical parameters of systems (7) - (9), conditions are obtained under which the solution of the problem blows up in finite time.

It is also noteworthy that in [11], system (7) was studied with the following boundary conditions:

$$-\left(u^n\right)_x(0,t)=u^{\alpha}\upsilon^{p}(0,t),\quad -\left(\upsilon^k\right)_x(0,t)=u^{q}\upsilon^{\beta}(0,t),\quad t>0.$$

The following equality is proven: $\min\{y_1-r_1,y_2-r_2\}=0,$ where

$$r_1=\frac{2p+k+1-2\beta}{4pq-(k+1-2\alpha)(n+1-2\beta)},$$

$$r_2=\frac{2p+n+1-2\beta}{4pq-(k+1-2\alpha)(n+1-2\beta)},$$

$$y_1=\frac{1-r_1(n-1)}{2},\quad y_2=\frac{1-r_2(k-1)}{2},$$

is a critical exponent of the Fujita type.

3 METHOD OF SOLUTION

This work is dedicated to studying the asymptotics of self-similar solutions to problem (1) - (3). For the case of slow diffusion ($m_1, m_2 > 1$), various self-similar solutions of problem (1) - (3) are constructed, which represent the asymptotics of the solutions to the problem under consideration. For numerical research, methods for selecting appropriate initial approximations for the iterative process that preserve the qualitative properties of problem (1) - (3) are proposed. An iterative process was developed, and numerical calculations were performed, demonstrating rapid convergence to the exact solution.

The system of equations (1) describes processes with finite propagation speed of disturbances when $m_i > 1 \ (i = 1,2)$. Equations (1) are degenerate at $u(x,t)$, $\upsilon(x,t) = 0$, therefore problem (1) - (3) admits a generalized solution that lacks the necessary smoothness at the degeneracy points.

The system (1) has compactly supported self-similar solutions of the form

$$\begin{cases} \underline{u}(x,t) = (T+t)^{-\alpha_1} f(\xi), \\ \underline{\upsilon}(x,t) = (T+t)^{-\alpha_2} \varphi(\xi), \end{cases} \tag{10}$$

where $\xi = |\zeta|$, $\zeta_1 = (x_1 + h_1)(T+t)^{-\beta}$, $\zeta_i = x_i (T+t)^{-\beta}$, $(i = 2,\dots,N)$, $T > 0$, $\beta = \dfrac{q_1 - m_2}{2q_1 - m_2 - 1} = \dfrac{q_2 - m_1}{2q_2 - m_1 - 1}$, $\alpha_1 = \dfrac{1}{2q_1 - m_2 - 1}$, $\alpha_2 = \dfrac{1}{2q_2 - m_1 - 1}$ va $\left(f(\xi), \varphi(\xi)\right)$ and are solutions to the following problems:

$$\begin{cases} \xi^{N-1} \dfrac{d}{d\xi}\left(\xi^{1-N} \varphi^{m_1 - 1} \dfrac{df}{d\xi}\right) + \beta\xi \dfrac{df}{d\xi} + \alpha_1 f = 0, \\ \xi^{N-1} \dfrac{d}{d\xi}\left(\xi^{1-N} f^{m_2 - 1} \dfrac{d\varphi}{d\xi}\right) + \beta\xi \dfrac{d\varphi}{d\xi} + \alpha_2 \varphi = 0, \end{cases} \tag{11}$$

$$\begin{cases} -\varphi^{m_1 - 1} \dfrac{df}{d\xi}(0) = f^{q_1}(0), \\ -f^{m_2 - 1} \dfrac{d\varphi}{d\xi}(0) = \varphi^{q_2}(0), \end{cases} \tag{12}$$

This is obtained by substituting (10) into (1) - (3) and performing some simplifications.

Let us consider the following functions:

$$\begin{cases} \tilde{f}(\xi) = \left(a - \xi^2\right)^{\frac{1}{m_2 - 1}}, \\ \tilde{\varphi}(\xi) = \left(a - \xi^2\right)^{\frac{1}{m_1 - 1}}, \end{cases} \tag{13}$$

where $a > 0$.

Theorem. Suppose $m_1 > 1$ and $m_2 > 1$, then the compactly supported solution of the system of equations (11) has $\xi \to \sqrt{a}$ the asymptotics

$$\begin{cases} f(\xi) = c_1 \tilde{f}(\xi)\left(1 + o(1)\right), \\ \varphi(\xi) \simeq c_2 \tilde{\varphi}(\xi)\left(1 + o(1)\right), \end{cases} \tag{14}$$

where $c_1 = \left(\dfrac{\beta(m_2 - 1)}{2}\right)^{1/(m_1 - 1)}$, $c_2 = \left(\dfrac{\beta(m_1 - 1)}{2}\right)^{1/(m_2 - 1)}$.

The theorem is proven as in [12].

4 RESULTS AND DISCUSSION

The numerical scheme is constructed based on the finite difference method. For this purpose, equations (1) are approximated with second-order accuracy in spatial coordinates and first-order accuracy in time. An iterative process is constructed, where node values are calculated using the expansion method at the inner iteration steps. It is known that in the general case, the main difficulty in numerically solving the nonlinear problem (1) - (3) is selecting an appropriate initial approximation for the iterative process.

When solving specific problems, functions that reflect certain properties of the sought solutions are used. These functions are obtained through qualitative analysis of the problem. This difficulty is overcome by successfully selecting initial approximations depending on the values of the numerical parameters in the equations. For this purpose, an established asymptotic formula is used in the calculations. Numerical calculations were performed based on the above results. Some results of the numerical schemes and computational experiments are presented below.

A computational experiment was conducted using the aforementioned numerical schemes. Some results of the numerical experiments are presented below. The grid step is very small, $h = 0.05$, the number of nodes is $N = 2500$, and the $\varepsilon = 10^{-3}$ iteration accuracy is specified. The calculation was carried out up to $t = 2$ with $\tau = 0.02$ time steps. Formulas (10) and (14) were used as the initial approximation for the iterative process.

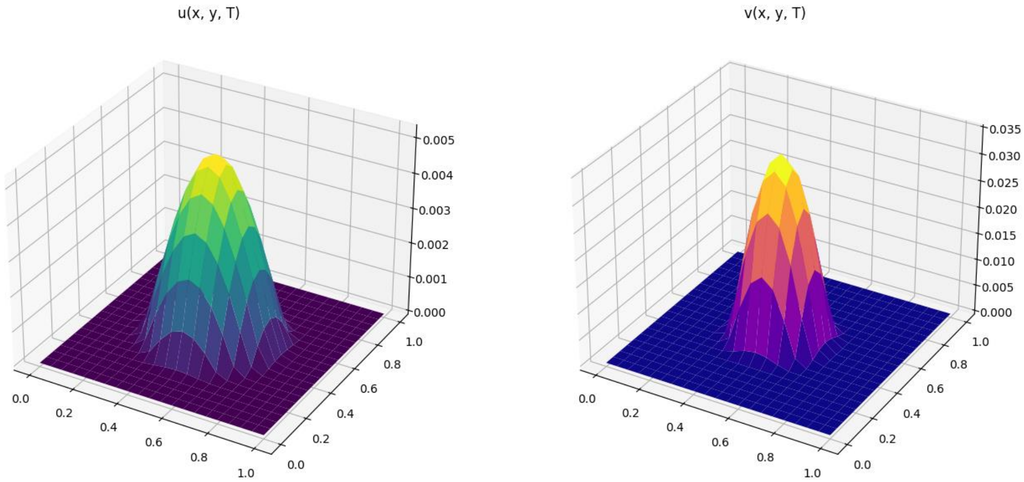

Fig. 1. The numerical solution of problem (1) - (3). $q_1 = 4.75,\ q_2 = 5.5,\ m_1 = 1.15,\ m_2 = 1.35$

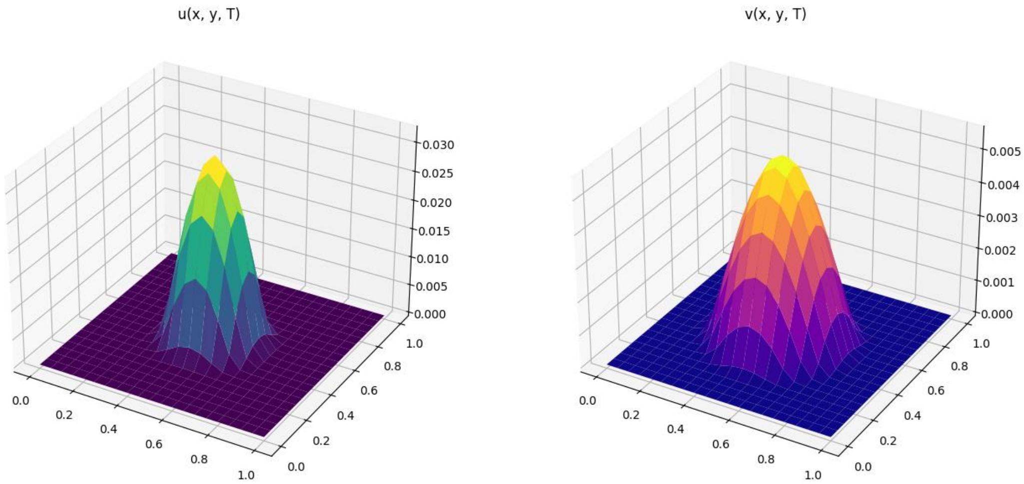

Fig. 2. The numerical solution of problem (1) - (3). $q_1 = 1.75,\ q_2 = 1.95,\ m_1 = 1.55,\ m_2 = 1.65$

Figures 1-3 show the results of the numerical solution of problem (1) - (3), corresponding to the case of slow diffusion. Asymptotic formulas $m_i > 1\ (i = 1, 2)$ (10), (14) and graphs demonstrate that the body moves at a finite velocity. The propagation depth of the diffusion wave depends on time and the wave front (the point where $\underline{u}(x,t)$, $\underline{\upsilon}(x,t)$ vanishes), and for each medium, it is located at the endpoint. The asymptotic formulas $x_\phi = \sqrt{a}\left(T+t\right)^{\beta} < \infty$, (10), (14) and graphs indicate that the body moves at a finite velocity.

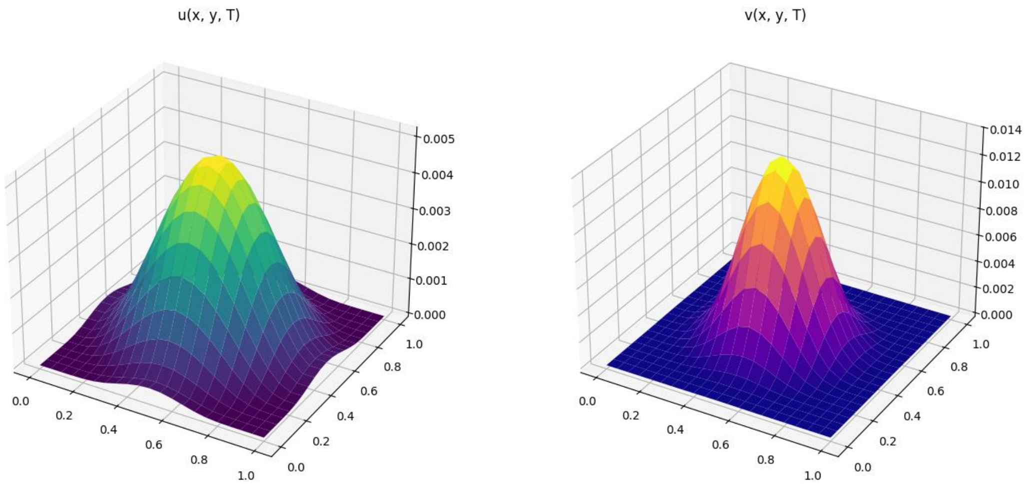

Fig. 3. The numerical solution of problem (1) - (3). $q_1 = 1.8,\ q_2 = 2.3,\ m_1 = 1.75,\ m_2 = 1.25$

5 CONCLUSION

The mathematical model presented in (1) - (3) is resolved by numerical methods and characterises the condition of slow diffusion. The asymptotic formulas (10) and (14), along with the derived findings, suggest that the substance or diffusive "particle" traverses the medium at a finite velocity. The depth of the diffusion wave is contingent upon time, with the wavefront coinciding with the boundary of each medium. These findings are particularly significant when analysing diffusion processes in complex media or multicomponent systems. Movement at a constrained velocity is a significant phenomenon for actual physical systems, applicable in cross-diffusion processes, the petroleum sector, biophysics (such as drug permeation into tissues), and materials research. It is particularly advantageous in assessing the local region of influence, such as in regulating the dissemination of hazardous materials or formulating selected transportation models.

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ЧИСЛЕННОЕ РЕШЕНИЕ МНОГОМЕРНОЙ ЗАДАЧИ КРОСС-ДИФФУЗИИ С НЕЛОКАЛЬНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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Аннотация. В данной статье исследуется асимптотическое поведение автомодобных решений нелинейной кросс-диффузионной системы с нелокальными граничными условиями. Определен главный член асимптотики автомодобных решений. Для численного исследования рассматриваемой задачи предложен метод выбора начального приближения, соответствующего итерационному процессу. Численные вычисления и анализ результатов были выполнены с использованием асимптотических формул в качестве начального приближения для процесса итерации.

Ключевые слова: кросс-диффузия, нелокальное граничное условие, автомодель.