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A COUPLED PROBLEM IN STRESSES ON LOADING A HOMOGENEOUS SEMI-INFINITE THERMOELASTIC ROD

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Abstract. *Usually, the complete system of thermoelasticity equations consists of the equation of motion, the Dugamel-Neumann relations, the Cauchy relation, and the heat influx equation. In this case, usually the boundary value problem is reduced to a system of differential equations for displacements and temperature with the corresponding initial and boundary conditions. Coupled thermoelasticity problems can be formulated in both stress and temperature, but this requires the second order of smoothness of the stress tensor. In this regard, there is an additional condition regarding stress. In this paper, the coupled problems of thermoelasticity are proposed to be considered on the basis of the equations of motion, the continuity equation, the Dugamel-Neumann relation, and the heat influx equation. In this case, a system of three interrelated equations for stress, velocity and temperature was obtained. Based on this system, the problem of the stress-strain state of a homogeneous semi-infinite thermoelastic rod under the action of a dynamic thermomechanical load applied to the end of the rod is considered. The exact analytical solution was obtained by the continuation method followed by the application of the Laplace and Fourier integral transformations. The asymptotic behavior of the stress is studied for small and large values of time. The propagation velocity of a thermoelastic wave has also been studied. Numerical calculations are carried out, the qualitative and quantitative nature of the mutual influence of temperature on the distribution of stresses and velocities of the rod sections is analyzed.*

Keywords: *stress, temperature, velocity, rod, coupled problem, Fourier and Laplace transforms.*

I. INTRODUCTION

The study of the process of deformation of structures and their elements, taking into account thermal and mechanical effects, is an urgent problem in the solid mechanics. Usually, boundary value problems describing the process of thermomechanical deformation consist of the equation of motion, the Dugamel–Neumann relation, the Cauchy relation, the heat influx equation with the corresponding initial and boundary conditions, and are usually called coupled problems of thermoelasticity. One of the options for taking into account the influence of heat is the use of the classical equation of heat conduction - an equation of parabolic type.

It should be noted that the classical heat equation based on the Fourier hypothesis about the proportionality of the heat flux to the temperature gradient leads to infinity of the propagation velocity of thermal perturbations, which contradicts the fundamental physical laws [1]. In this regard, over the past three decades, various models have been proposed that are described by a hyperbolic type equation or an equation of a higher order [6, 7, 11, 24, 30, 31]. As noted in [30, 34], the equations of heat propagation of the hyperbolic type also generate paradoxes (violation of the law of conservation of energy). The paradox of solving the classical heat equation is of a qualitative nature. In fact, at any finite time in some finite region near the heat source, a practically significant change in temperature occurs.

For example, if there is an instantaneous heating of the end of a semi-infinite rod, then for any fixed value of time there is a finite distance at which the temperature will differ little from the initial one with the required accuracy. Indeed, in this case, the temperature distribution $T(x,t)$ is described by the solution of the following problem:

$$\frac{\partial T}{\partial t} = a_T{}^2 \frac{\partial^2 T}{\partial x^2}, 0 < x < \infty,\ t > 0,$$

$$T(x,0) = T_0 = const.\ , T(0,t) = T_1 = const., T(x,t)\big|_{t\to\infty} = T_0,$$

where $a_T{}^2$ – temperature coefficient of conductivity. As is known, the solution to this problem has the form

$$T(x,t) = T_0 + (T_1 - T_0) erfc\ (\frac{x}{2a_T\sqrt{t}}),$$

$$erfcz = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\xi^2} d\xi.$$

Let the permissible error be no more than 0.1 percent, then in the area where the inequality is satisfied

$$\frac{T(x,t) - T_0}{T_0} = \frac{T_1 - T_0}{T_0} erfc(\frac{x}{2a_T\sqrt{t}}) \le 0.001,\ erfc(\frac{x}{2a_T\sqrt{t}}) \le 0.001 \frac{T_0}{T_1 - T_0}.$$

can be calculated with the accepted accuracy $T(x,t) = T_0$. In this case, the curves on which the equality holds can be considered as a heat propagation front. According to the theory of linear thermoelasticity $\frac{T_1 - T_0}{T_0} << 1$. Therefore, it can be assumed that $erfc(\frac{x}{2a_T\sqrt{t}}) \le 0.01$.

Equality holds if $\frac{x}{2a_T\sqrt{t}} = 1.8214$ because $erfcz$ is a decreasing function and $erfc(1.8214) = 0.01$. Therefore, with an allowable error $x = 3.64284 a_T\sqrt{t}$, the results of the calculations performed in this work (Fig. 4, Fig. 5) can also be taken as the heat propagation front.

To date, there are quite a few works devoted to various problems of thermoelasticity based on one or another model of heat influx. Coupled problems of thermoelasticity within the framework of thermodynamic laws on displacements and temperature were considered in the works of M. Biot [1], V. Novatsky [2], I.K. Shevchenko [3], Karnaukhova [4] and others [5-8].

Analytical solutions of thermoelastic problems of the generalized theory of thermoelasticity for semi-infinite regions were obtained in the following works [8-12]. Numerical methods for solving coupled thermoelastic boundary value problems based on finite difference methods are considered in [15,16,21]. The papers [32, 33] studied the thermo-mechanical state of a functionally graded hollow sphere and disk under the action of thermomechanical loads. The analytical solution obtained using the Laplace transform is compared with the numerical results of the finite element method. In [20], an analytical solution of a plane thermoelasticity problem for an orthotropic half-plane was constructed by the method of integral transformations. Questions of variational formulations of coupled problems, as well as taking into account destructions, were studied in [13, 14]. The following works [17–19] are also devoted to the numerical solution of one-dimensional and two-dimensional coupled dynamic problems of thermoelasticity.

In this paper, within the framework of linear thermoelasticity, we consider the stress-strain state of a semi-infinite elastic rod caused by a dynamic load applied to the end of the rod and a change in temperature at this end. The problem is posed in stresses and its analytical solution is found by the method of Laplace and Fourier integral transformations. The asymptotic behavior of stress and velocity is studied for small and large values of time. The propagation velocity of a thermoelastic wave is revealed. Numerical calculations are made, the qualitative and quantitative nature of the mutual influence of stress and temperature is analyzed.

The work consists of six sections. In the second section, based on the equations of linear thermoelasticity - the equation of motion, the continuity equation for the case of small deformations, the Dugamel-Neumann relation and the heat influx equation, a system of three interrelated equations is obtained for stress, rod cross-section velocity and temperature.

In the third section, the problem statement is given. The solution method is described: the method of continuation with the subsequent application of the Laplace and Fourier integral transformations. The form of the desired functions is in the area of the Laplace and Fourier transforms.

In the fourth section, the desired functions are found - stresses, velocities and temperatures by performing the inverse Laplace and Fourier transforms. The asymptotic behavior of the solution for small and large values of time is studied. Analytical solutions of the problem are obtained for the case of sudden loading and instantaneous heating, with the help of which solutions are also found for the general case.

In the fifth section, the case of simultaneous sudden loading and instantaneous heating of the end of the rod is considered in detail. A qualitative and quantitative analysis of the mutual influence of stress and temperature was carried out both by appropriate estimates and by analyzing the results of numerical calculations, which are presented in the form of graphs and tables.

II. BASIC EQUATIONS AND RELATIONS

Consider the equation of motion for uniaxial compression of the rod [22-25]:

$$\rho\frac{\partial v}{\partial t}=\frac{\partial \sigma}{\partial x} \tag{1}$$

where $v(x,t)-$velocity, $\sigma=\sigma_{11}(x,t)-$ stress, $\rho-$density. In the case of small strains, it can be found from the continuity equation [23] that:

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial t}I_1(\varepsilon) \tag{2}$$

where $I_1(\varepsilon)-$ the first invariant of the strain tensor.

The constitutive relation of the linear thermoelasticity of the Dugamel-Neumann law has the form [2, 23]

$$\sigma_{ij} = \lambda I_1(\varepsilon)\delta_{ij} + 2\mu\varepsilon_{ij} - \\ -(3\lambda+2\mu)\alpha(T-T_0)\delta_{ij}, \tag{3}$$

or inverted

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \\ -\nu I_1(\sigma)\delta_{ij}] + \alpha(T-T_0)\delta_{ij} \tag{4}$$

where λ and μ Lame parameters, $\delta_{ij} -$ Kronecker symbol, $\varepsilon_{ij} -$ strain tensor components; $I_1(\varepsilon)$ and $I_1(\sigma)$ first invariants of strain tensor and stress tensor, $E -$ Young's modulus, $\nu -$ Poisson's ratio, $\alpha -$ linear expansion coefficient, $T -$ absolute temperature, $T_0 -$ initial temperature value.

If, in the considered case of uniaxial compression, we assume that:

$$\sigma_{11} = \sigma, \quad \sigma_{22} = \sigma_{33} = 0, \\ \varepsilon_{11} = \varepsilon, \quad \varepsilon_{22} = \varepsilon_{33} = 0, \tag{5}$$

then (3) takes the form:

$$\sigma = (\lambda+2\mu)\varepsilon - 3\alpha K\theta \tag{6}$$

where $\theta = T - T_0$, $K = \lambda + \frac{2}{3}\mu$ - volumetric compression module. From here:

$$\varepsilon = \frac{\sigma + 3\alpha K\theta}{\lambda+2\mu} = I_1(\varepsilon). \tag{7}$$

Thus, if we neglect the lateral deformation that occurs both under the action of stress and due to temperature changes, i.e., we assume that stress and strain, as well as temperature, change only along the axis of the rod, then Hooke's law and its inversion, respectively, have the form (6) and (7). Within the framework of one-dimensional motion, to a certain extent, the influence of deformation in the lateral direction can be taken into account if we consider the parameters averaged over the sections of the rod (displacements, stresses, etc.), for this we will assume that $\sigma_{11} = \sigma, \sigma_{22} = \sigma_{33} = 0$ and use the reversed Hooke's law in the form (4). Then we have

$$\varepsilon_{11} = \frac{1}{E}[(1+\nu)\sigma - \nu I_1(\sigma)] + \alpha\theta = \frac{1}{E}\sigma + \alpha\theta, \\ \varepsilon_{22} = -\frac{\nu}{E}\sigma + \alpha\theta, \\ \varepsilon_{33} = -\frac{\nu}{E}\sigma + \alpha\theta, \tag{8}$$

consequently,

$$I_1(\varepsilon) = \frac{1}{E}\sigma(1-2\nu) + 3\alpha\theta. \tag{9}$$

Substituting (9) into (3), we obtain:

$$\sigma\left(1 - \frac{\lambda}{E}(1-2\nu)\right) = 3\lambda\alpha\theta + 2\mu\varepsilon - 3K\alpha\theta,$$

since there are equalities

$$1 - \frac{\lambda}{E}(1-2\nu) = \frac{1}{1+\nu}, \\ 2\mu = \frac{E}{1+\nu}, \quad \lambda - K = -\frac{2}{3}\mu,$$

(3) takes the form

$$\sigma = E(\varepsilon - \alpha\theta). \tag{10}$$

Assuming that the Fourier law takes place, the heat influx equation is written as [1-3]:

$$\rho c_v \frac{\partial\theta}{\partial t} = k\cdot\Delta\theta - (3\lambda+2\mu)\alpha T\frac{\partial}{\partial t}I_1(\varepsilon), \\ \theta = T - T_0, \tag{11}$$

where $c_v -$ specific heat, $k -$ coefficient of thermal conductivity.

The system of equations (1), (2) and (11), taking into account (9) (or (7)) forms a complete system of equations for three functions σ, v, θ. This system for small temperature changes $(\frac{\theta}{T_0} << 1)$ after substituting (9) into equations (2) and (11) takes the form:

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x} \qquad (12)$$

$$\frac{\partial v}{\partial x} = \frac{1}{E}(\alpha_1 \frac{\partial \sigma}{\partial t} + 3\alpha E \frac{\partial \theta}{\partial t}), \qquad (13)$$

$$\frac{\partial \theta}{\partial t} = a_T^2 \frac{\partial^2 \theta}{\partial x^2} - \frac{\alpha_1 \alpha_2}{E} \frac{\partial \sigma}{\partial t}, \qquad (14)$$

where

$$\alpha_1 = 1 - 2\nu,$$

$$\alpha_2 = \frac{3\alpha T_0 K}{\rho c_v + 9\alpha^2 K T_0},$$

$$a_T^2 = \frac{k}{\rho c_v}.$$

III. PROBLEM STATEMENT AND SOLUTION METHOD

Formulation of the problem. It is required to find a solution to the system of equations (12) - (14) that satisfies the following initial

$$\sigma(x,t)\big|_{t=0} = 0,$$

$$v(x,t)\big|_{t=0} = 0, \qquad (15)$$

$$0 < x < \infty,$$

$$\theta(x,t)\big|_{t=0} = 0,\ 0 < x < \infty \qquad (16)$$

and boundary conditions

$$\sigma(x,t)\big|_{x=0} = -\sigma_0(t), \quad t > 0, \ \ (\sigma_0(t) > 0), \qquad (17)$$

$$\sigma(x,t)\big|_{x\to\infty} = 0, \ \ t > 0, \qquad (18)$$

$$\theta(x,t)\big|_{x=0} = \theta_0(t), \ \ t > 0, \qquad (19)$$

$$\theta(x,t)\big|_{x\to\infty} = 0, \ \ t > 0. \qquad (20)$$

Using the continuation method [26], we solve the formulated problem by successively applying the Laplace transform and the Fourier transform [27, 28]:

$$F(p) = \int_0^\infty f(t) e^{-pt} dt,$$

$$\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x) e^{-i\lambda x} dx, \qquad (21)$$

where p - point in the complex plane. These transformations will be written in symbolic form, respectively

$$F(p) = \hat{f} \to f, \quad \Phi(\lambda) = \tilde{\varphi} \to \varphi \qquad (22)$$

where the upper sign ^ means the Laplace transform, and the upper sign ~ means the Fourier transform. In the considered problem (12)-(14), (15)-(20) it is assumed that the desired functions $\sigma(x,t)$, $v(x,t)$, $\theta(x,t)$ are defined for $t \geq 0$ on the half-line $x > 0$. Let us extend these functions to the half-line $x < 0$ by odd and even extensions as follows [26]:

$$\Sigma = \begin{cases} \sigma(x,t), x > 0, \\ -\sigma(-x,t), x < 0, \end{cases}$$

$$V = \begin{cases} v(x,t), x > 0, \\ v(-x,t), x < 0, \end{cases} \qquad (23)$$

$$\Theta = \begin{cases} \theta(x,t), x > 0, \\ -\theta(-x,t), x < 0. \end{cases}$$

At the same time, according to the continuation method, instead of the problem (12)-(14), (15)-(20), we obtain the following problem for the functions $\Sigma(x,t)$, $V(x,t)$, $\Theta(x,t)$: find in the region $\{x,t: -\infty < x < \infty,\ t > 0\}$ the solution of the following system of equations

$$\rho \frac{\partial V}{\partial t} = \frac{\partial \Sigma}{\partial x}, \qquad (24)$$

$$\frac{\partial V}{\partial x} = \frac{1}{E}(\alpha_1 \frac{\partial \Sigma}{\partial t} + 3\alpha E \frac{\partial \Theta}{\partial t}), \qquad (25)$$

$$\frac{\partial \Theta}{\partial t} = a_T^2 \cdot \frac{\partial^2 \Theta}{\partial x^2} - \frac{\alpha_1 \alpha_2}{E} \frac{\partial \Sigma}{\partial t}, \qquad (26)$$

satisfying the conditions

$$\begin{aligned} &\Sigma \to 0,\ V \to 0,\ \Theta \to 0 \\ &for \quad t \to 0\ u\ |x| \to \infty \end{aligned} \qquad (27)$$

$$\begin{aligned} &\Sigma\big|_{x=+0} - \Sigma\big|_{x=-0} = -2\sigma(0,t), \\ &V\big|_{x=+0} - V\big|_{x=-0} = 0, \\ &\Theta\big|_{x=+0} - \Theta\big|_{x=-0} = 2\theta(0,t), \quad t > 0. \end{aligned} \qquad (28)$$

Applying successively the Laplace transform and the Fourier transform to equations (24) - (26) taking into account conditions (27) - (28), we obtain

$$\rho p\tilde{\hat{V}} = -i\lambda\tilde{\hat{\Sigma}} - \sqrt{\frac{2}{\pi}}\,\hat{\sigma}(0,t) \qquad (29)$$

$$-i\lambda\tilde{\hat{V}} = \frac{p}{E}(\alpha_1\tilde{\hat{\Sigma}} + 3\alpha E\tilde{\hat{\Theta}}), \qquad (30)$$

$$\begin{aligned} &(p + a_T{}^2\lambda^2)\tilde{\hat{\Theta}} = -p\frac{\alpha_1\alpha_2}{E}\tilde{\hat{\Sigma}} + \\ &+i\sqrt{\frac{2}{\pi}}\,a_T{}^2\,\lambda\hat{\theta}(0,t), \end{aligned} \qquad (31)$$

by the method of elimination from (29)-(31) we find that the $\tilde{\hat{\Sigma}}-$ stress image Σ has the form

$$\tilde{\hat{\Sigma}} = \sqrt{\frac{2}{\pi}}\frac{a^2}{\alpha_1}\frac{i\lambda\hat{\sigma}(0,t) - 3\alpha\rho a_T{}^2\dfrac{p^2}{p + a_T{}^2\lambda^2}i\lambda\hat{\theta}(0,t)}{p^2 + \lambda^2\dfrac{a^2}{\alpha_1} - 3\alpha\alpha_2\dfrac{p^3}{p + a_T{}^2\lambda^2}},\ \ a^2 = \frac{E}{\rho}. \qquad (32)$$

The functions $\tilde{\hat{V}}$ and $\tilde{\hat{\Theta}}$ are obtained by substituting (32) into (30) and (31). Further, performing the inverse Laplace and Fourier transforms, we find the functions $\Sigma(x,t),\ V(x,t),\ \Theta(x,t)$, which, for x>0, are solutions to the problem.

IV. SOLUTION OF THE PROBLEM

Asymptotics of the solution for small and large values of time.

As is known [27], small values of t correspond to large values of the parameter p of the Laplace transform, and large values of t correspond to small values of the parameter p. Taking advantage of this, we first consider the asymptotics of the voltage for small and large values of the time t.

a) ***Small values of time***. The denominator in (32) can be represented as

$$\begin{aligned} &p^2 + \lambda^2\frac{a^2}{\alpha_1} - 3\alpha\alpha_2\frac{p^3}{p + a_T{}^2\lambda^2} = \\ &= p^2(1 - 3\alpha\alpha_2\frac{p}{p + a_T{}^2\lambda^2}) + \lambda^2\frac{a^2}{\alpha_1}, \end{aligned}$$

where

$$\begin{aligned} &1 - 3\alpha\alpha_2\frac{p}{p + a_T{}^2\lambda^2} = \\ &= 1 - 3\alpha\alpha_2(1 - \frac{a_T{}^2\lambda^2}{p + a_T{}^2\lambda^2}) \approx 1 - 3\alpha\alpha_2. \end{aligned}$$

that's why

$$\begin{aligned} &p^2 + \lambda^2\frac{a^2}{\alpha_1} - 3\alpha\alpha_2\frac{p^3}{p + a_T{}^2\lambda^2} \approx \\ &\approx p^2(1 - 3\alpha\alpha_2) + \lambda^2\frac{a^2}{\alpha_1} = \\ &= (1 - 3\alpha\alpha_2)(\mathrm{p}^2 + \lambda^2\frac{a^2}{\alpha_1(1 - 3\alpha\alpha_2)}). \end{aligned}$$

Consequently, (32) takes the form

$$\tilde{\hat{\Sigma}} = -\sqrt{\frac{2}{\pi}}\,a_*{}^2\,\frac{i\lambda\hat{\sigma}_0(t) + 3\alpha\rho a_T{}^2\dfrac{p^2}{p + a_T{}^2\lambda^2}i\lambda\hat{\theta}_0(t)}{p^2 + \lambda^2\,a_*{}^2} \qquad (33)$$

where

$$\hat{\sigma}_0(t) = -\hat{\sigma}(0,t),$$

$$\hat{\theta}_0(t) = \hat{\theta}(0,t),$$

$$a_*^{\,2} = \frac{a^2}{\alpha_1(1-3\alpha\alpha_2)}.$$

b) ***Large time values***. In this case, the approximate equality

$$p^2 + \lambda^2\frac{a^2}{\alpha_1} - 3\alpha\alpha_2\frac{p^3}{p + a_T^{\,2}\lambda^2} =$$

$$= p^2(1-3\alpha\alpha_2) + 3\alpha\alpha_2 p^2\left(1 - \frac{p}{p + a_T^{\,2}\lambda^2}\right) + \lambda^2\frac{a^2}{\alpha_1} \approx$$

$$\approx p^2(1-3\alpha\alpha_2) + +\lambda^2\frac{a^2}{\alpha_1}$$

From here and from (33) it follows that both for small and for large values of t, the stress image $\tilde{\hat{\Sigma}}$ has the form

$$\tilde{\hat{\Sigma}}(\lambda, p) = \tilde{\hat{A}}_1(\lambda, p; \sigma_0) - \\ -\sqrt{\frac{2}{\pi}}\,3\alpha\rho a_T^2 a_*^2\,\tilde{\hat{A}}_2(\lambda, p; \theta_0) \qquad (34)$$

where

$$\tilde{\hat{A}}_1(\lambda, p; \sigma_0) = -\sqrt{\frac{2}{\pi}}a_*^2\frac{i\lambda\hat{\sigma}_0(t)}{p^2 + a_*^2\lambda^2}, \\ \tilde{\hat{A}}_2(\lambda, p; \theta_0) = \frac{a_T^2 p^2}{p + a_T^2\lambda^2}\cdot\frac{i\lambda\hat{\theta}_0(t)}{p^2 + a_*^2\lambda^2} \qquad (35)$$

according to the formula for inverting the Fourier transform [28], we find:

$$\hat{A}_1(x, p; \sigma_0) = -\sqrt{\frac{2}{\pi}}\hat{\sigma}_0(t)\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{+\infty}\frac{i\lambda\exp(-i\lambda x)}{\lambda^2 + \dfrac{p^2}{a_*^2}}d\lambda = \\ = -\hat{\sigma}_0(t)\frac{2}{\pi}\int_0^{+\infty}\frac{\lambda\sin(\lambda x)}{\lambda^2 + \dfrac{p^2}{a_*^2}}d\lambda = -\hat{\sigma}_0(t)\exp\left(-|x|\frac{p}{a_*}\right), \qquad (36)$$

because it is easy to show

$$\frac{\lambda\sin(\lambda x)}{\lambda^2 + \dfrac{p^2}{a_*^2}} \to \exp\left(-|x|\frac{p}{a_*}\right).$$

According to the delay theorem [27], we have the correspondence

$$\hat{\sigma}_0(t)\exp\left(-x\frac{p}{a_*}\right) \to \sigma_0\left(t - \frac{x}{a_*}\right)H\left(t - \frac{x}{a_*}\right),$$

$$x > 0$$

where $H(t) -$ Heaviside function. According to (35)

$$A_1(x,t;\sigma_0) = -\sigma_0\left(t - \frac{x}{a_*}\right)H\left(t - \frac{x}{a_*}\right), \qquad (37)$$

in particular, if $\sigma_0(\mathrm{t}) = \sigma_{0c} = const$

$$\sigma_{0c} \to \hat{\sigma}_{0c}(\mathrm{p}) = \frac{\sigma_{0c}}{p},$$

$$A_1(x,t;\sigma_{0c}) = -\sigma_{0c}\,H\left(t - \frac{x}{a_*}\right), \qquad (38)$$

$$x > 0$$

The original function $\tilde{\hat{A}}_2(\lambda, \mathrm{p}; \theta_0)$ find first for a special case $\theta_0(t) = \theta_{0c} = const.$

$$\tilde{\hat{A}}_2(\lambda, p; \theta_{0c}) = a_T^2\frac{p}{p + a_T^2\lambda^2}\cdot \\ \cdot\frac{i\lambda\theta_{0c}}{p^2 + a_*^2\lambda^2} \to \tilde{A}_2(\lambda, t; \theta_{0c}).$$

Using the table correspondences of each factor and the multiplication theorem, we obtain the following correspondence:

$$\frac{p}{p + a_T^2\lambda^2}\cdot\frac{1}{p^2 + a_*^2\lambda^2} \to \\ \to \frac{\lambda^2 a_T^2}{\lambda^4 a_T^4 + \lambda^2 a_*^2}[\cos(\lambda a_* t) - \\ -\exp(-\lambda^2 a_T^2 t) + \frac{a_*}{\lambda a_T^2}\sin(\lambda a_*\,\mathrm{t})]$$

consequently

$$\tilde{A}_2(\lambda,t;\theta_{0c}) = i\lambda\theta_{0c}\frac{\lambda^2 a_T^4}{\lambda^4 a_T^4+\lambda^2 a_*^2}[\cos(\lambda a_* t)-\exp(-\lambda^2 a_T^2 t)+\frac{a_*}{\lambda a_T^2}\sin(\lambda a_* t)]$$

from here, by the formula for inverting the Fourier transform, we find

$$A_2(x,t;\theta_{0c}) = \frac{2}{\pi}\theta_{0c}\int_0^\infty \frac{\lambda(\cos(\lambda a_* t)-\exp(-\lambda^2 a_T^2 t))+\frac{a_*}{a_T^2}\sin(\lambda a_* \text{ t})}{\lambda^2+\frac{a_*^2}{a_T^4}}\sin(\lambda x)d\lambda \qquad (39)$$

Thus, as follows from (33), (37) and (39), the stresses at small and large values of t in the case of $\theta_0(\text{t}) = \theta_{0c} = const$ has the form

$$\sigma(x,t) = -\sigma_0(t-\frac{x}{a_*})\text{H}(t-\frac{x}{a_*})-\sigma_\theta(x,t;\theta_{0c}),$$

$$\sigma_\theta(x,t;\theta_{0c}) = 3\alpha\theta_{0c}\rho a_*{}^2\frac{2}{\pi}\int_0^\infty \frac{\lambda(\cos(\lambda a_* t)-\exp(-\lambda^2 a_T^2 t))+\frac{a_*}{a_T{}^2}\sin(\lambda a_* t)}{\lambda^2+\frac{a_*^2}{a_T^4}}\sin(\lambda x)d\lambda \qquad (40)$$

generally

$$\tilde{\hat{A}}_2(\lambda,\text{p};\theta_0(p)) = \frac{\theta_0(p)}{\theta_{0c}}p\tilde{\hat{A}}_2(\lambda,\text{p};\theta_{0c})\to$$

$$\to\hat{A}_2(x,p;\theta_0(p)) = \frac{\theta_0(p)}{\theta_{0c}}p\hat{A}_2(x,p;\theta_{0c})$$

where $\hat{A}_2(x,p;\theta_{0c})$ the Laplace transform of the right side (39).

Hence, since $\hat{A}_2(x,p;\theta_0(p))\to A_2(x,t;\theta_0(t))$, with the help of the multiplication theorem and the differentiation rule, the original function [27], we obtain

$$A_2(x,t;\theta_0(t)) = \frac{1}{\theta_{0c}}\theta_0(t)\otimes\frac{\partial}{\partial t}A_2(x,t;\theta_{0c}) =$$

$$= \frac{2}{\pi}\int_0^t\theta_0(t-\tau)\int_0^\infty \frac{\lambda(\cos(\lambda a_*\tau)-\exp(-\lambda^2 a_T^2\tau))+\frac{a_*}{a_T^2}\sin(\lambda a_*\tau)}{\lambda^2+\frac{a_*^2}{a_T^4}}\sin(\lambda x)d\lambda d\tau \qquad (41)$$

where $\otimes-$ means function convolution. Thus, from (34), (37) and (41) it follows that the stresses in the general case has the form

$$\begin{cases}\sigma(x,t) = -\sigma_0(t-\frac{x}{a_*})\text{H}(t-\frac{x}{a_*})-\sigma_\theta(x,t;\theta_0), \quad 0\le x<\infty,\ \text{t}\ge 0\\ \sigma_\theta(x,t;\theta_0) = 3\alpha\rho a_*{}^2 A_2(x,t;\theta_0)\end{cases} \qquad (42)$$

where $\sigma_\theta(x,t;\theta_0)-$ describes the effect of temperature on stress distribution; $A_2(x,t;\theta_0)$ is determined by formula (41). Formula (42) for the stress was obtained within the framework of one-dimensional motion, taking into account the transverse deformation (stresses, velocities, etc. are considered to be averaged over the cross section of the rod). In this case, the propagation velocity of the thermoelastic wave differs from the propagation velocity of the elastic wave $a=\sqrt{E/\rho}$ and has the form

$$a_* = a\sqrt{\frac{1}{\alpha_1(1-3\alpha\alpha_2)}} \qquad (43)$$

here, the effect of lateral deformation is taken into account by parameter α_1, and the effect of temperature change by parameters α and α_2. In this case, the effect of temperature in quantitative terms is very insignificant, because

$$3\alpha\alpha_2 = \frac{9\alpha^2 T_0 K}{\rho c_v + 9\alpha^2 K T_0} << 1.$$

If we neglect the lateral deformation and assume that the parameters of the problem change only along the rod, and the constitutive relation has the form (6), then the stress is determined by the formula

$$\begin{gathered}\sigma(x,t) = -\sigma_0\left(t-\frac{x}{c}\right)\mathrm{H}\left(t-\frac{x}{c}\right)-\sigma_\theta(x,t;\theta_0), \quad 0\le x<\infty, \mathrm{t}\ge 0;\\ \sigma_\theta(x,t;\theta_0)=\alpha(\lambda+2\mu)A_2(x,t;\theta_0),\end{gathered} \qquad (44)$$

here $A_2(x,t;\theta_0)$ is determined by formula (41), in which a_* is replaced by

$$c=\sqrt{\frac{\lambda+2\mu}{\rho(1-3\alpha\,\alpha_2)}},$$

$$\alpha_2 = \frac{3\alpha T_0 K}{\rho c_v + 3\alpha^2 K T_0}$$

where $c-$ wave propagation velocity, which, apparently, differs little from the longitudinal wave propagation velocity.

The validity of formula (43) can be verified by the method of characteristics. The equations for determining the characteristics have the form

$$\begin{gathered}\rho\frac{\partial v}{\partial t}-\frac{\partial\sigma}{\partial x}=0,\\ \frac{\partial v}{\partial x}-\frac{\alpha_1}{E}(1-3\alpha\alpha_2)\frac{\partial\sigma}{\partial t}=f,\\ \frac{\partial v}{\partial t}dt+\frac{\partial v}{\partial x}dx=0\\ \frac{\partial\sigma}{\partial t}dt+\frac{\partial\sigma}{\partial x}dx=0\end{gathered} \qquad (45)$$

where it is taken into account that in equation (13) $\frac{\partial}{\partial t}\mathrm{I}_1(\varepsilon)$ can be represented as

$$\begin{gathered}\frac{\partial}{\partial t}\mathrm{I}_1(\varepsilon)=\frac{\alpha_1}{E}(1-3\alpha\alpha_2)\frac{\partial\sigma}{\partial t}+f;\\ f=3\alpha\, a_T{}^2\frac{\partial^2\theta}{\partial x^2}.\end{gathered} \qquad (46)$$

The characteristic equation of system (45) has the form

$$\begin{vmatrix}\rho & 0 & 0 & -1\\ 0 & 1 & -\frac{\alpha_1}{E}(1-3\alpha\alpha_2) & 0\\ dt & dx & 0 & 0\\ 0 & 0 & dt & dx\end{vmatrix}=0.$$

Calculating the determinant, we obtain the equations of characteristics

$$\begin{gathered}\left(\frac{dx}{dt}\right)^2=\frac{1}{\rho\frac{\alpha_1}{E}(1-3\alpha\alpha_2)},\\ \frac{dx}{dt}=\pm\sqrt{\frac{E}{\rho\alpha_1(1-3\alpha\alpha_2)}}.\end{gathered} \qquad (47)$$

where, as follows from (43), (46) and (47)

$$\frac{dx}{dt}=\pm a\sqrt{\frac{1}{\alpha_1(1-3\alpha\alpha_2)}}=\pm a_*.$$

The solution of equation (14) under conditions (16), (19) and (20) has the form

$$\begin{aligned}\theta(x,t)&=\int_0^t\theta_0(\tau)\frac{x}{2\sqrt{\pi a_T^2(t-\tau)^3}}\exp(-\frac{x^2}{4a_T^2(t-\tau)})\,d\tau-\\&-\frac{\alpha_1\alpha_2}{E}\int_0^t\int_0^\infty\sigma(\xi,\tau)\frac{\partial}{\partial t}G_1(x,\xi;t-\tau)d\xi d\tau\end{aligned}\qquad(48)$$

where $\sigma(x,t)$ is determined by formula (42), $G(x,\xi;t-\tau)-$ Green's function of the first boundary value problem for the heat equation. Note that (48) can also be obtained from (31) using the formulas for inverting the Fourier and Laplace transforms. The first term in (48) is the solution of the homogeneous heat equation that satisfies the initial condition and the boundary condition in the cross section x=0.

The second term gives the change in temperature caused by the change in stresses. The velocity of the sections of the rod in the image area according to (30) has the form

$$\tilde{\hat{V}}=\frac{i}{\lambda}\frac{p}{E}(\alpha_1\tilde{\hat{\Sigma}}+3\alpha E\tilde{\hat{\Theta}}),$$

from here, by the inversion formulas, using some properties of the Laplace and Fourier transforms, we find an expression for the velocity of the sections, which can be represented as

$$\begin{aligned}v(x,t)&=V_1(x,t;\sigma)+\\&+V_2(x,t;\sigma,\theta)+V_3(x,t;\theta),\end{aligned}\qquad(49)$$

where

$$\begin{aligned}V_1(x,t;\sigma)&=-\frac{\sigma_{0c}}{\rho a_*}H(t-\frac{x}{a_*}),\\V_2(x,t;\sigma,\theta)&=3\alpha\theta_{0c}a_*\frac{2}{\pi}\int_0^\infty\frac{\lambda((\cos(\lambda a_*\,\mathrm{t})+\exp(-\lambda^2a_T^2\,\mathrm{t}))+\dfrac{a_*}{\lambda a_T^2}\sin(\lambda a_*\,\mathrm{t})}{\lambda^2+\dfrac{a_*^2}{a_T^4}}\cos(\lambda x)d\lambda\\&+3\alpha\alpha_1\alpha_2\frac{\sigma_{0c}}{E}a_*\frac{2}{\pi}\int_0^\infty\frac{\cos(\lambda a_*\,\mathrm{t})-\exp(-\lambda^2a_T^2\,\mathrm{t}))+\dfrac{a_*}{\lambda a_T^2}\sin(\lambda a_*\,\mathrm{t})}{\lambda^2+\dfrac{a_*^2}{a_T^4}}\cos(\lambda x)d\lambda\\V_3(x,t;\theta)&=-3\frac{\alpha\theta_{0c}}{\sqrt{\pi t}}a_T\exp(-\frac{x^2}{4a_T^2t}).\end{aligned}\qquad(50)$$

Functions $V_1(x,t;\sigma),V_2(x,t;\sigma,\theta)$ and $V_3(x,t;\theta)$, on the right side of (49) describe the contribution to the value of the speed of the sections of the rod, respectively, from:

- sudden loading;

-temperature changes under the associated influence of stress and sudden heating;

-change in temperature by conduction during sudden heating.

In the case of boundary conditions (17) and (19), the formula for the velocity can be easily obtained similarly to (41)

$$\varepsilon = \frac{\sigma}{E} + \alpha\theta \qquad (51)$$

where $\sigma(x,t)$ is determined by formula (42) or, in a particular case, $\theta_0(t) = \theta_{0c} = const$ by formula (40), and $\theta(x,t) -$ by formula (48). Thus, solutions of problem (12)-14), (15)-(20) (i.e., functions $\sigma(x,t)$, $v(x,t)$ u $\theta(x,t)$), valid for small and large values of time, are obtained in the form (42), (48) and (49).

Consider the solution of the problem for an arbitrary value of time, assuming that $3\alpha\alpha_2 << 1$. In this case, there is an approximate equality

$$p^2 + \lambda^2 \frac{a^2}{\alpha_1} - 3\alpha\alpha_2 \frac{p^3}{p + a_T^2\lambda^2} =$$

$$= [(p^2 + \lambda^2 \frac{a^2}{\alpha_1})(p + a_T^2\lambda^2) - 3\alpha\alpha_2 p^3](p + a_T^2\lambda^2)^{-1} =$$

$$= \frac{p^3(1 - 3\alpha\alpha_2) + p^2 a_T^2\lambda^2 + p\lambda^2 \frac{a^2}{\alpha_1} + \lambda^4 \frac{a^2}{\alpha_1} a_T{}^2}{(p + a_T^2\lambda^2)} \approx$$

$$\approx \frac{p^3 + p^2 a_T^2\lambda^2 + p\lambda^2 \frac{a^2}{\alpha_1} + \lambda^4 \frac{a^2}{\alpha_1} a_T{}^2}{(p + a_T^2\lambda^2)} = p^2 + \lambda^2 \frac{a^2}{\alpha_1}$$

in this case, the stress image (32) takes the form

$$\tilde{\hat{\Sigma}} = \sqrt{\frac{2}{\pi}} \frac{a^2}{\alpha_1} \frac{i\lambda\hat{\sigma}_0(\mathrm{t}) - 3\alpha\rho a_T^2 \frac{p^2}{p + a_T^2\lambda^2} i\lambda\hat{\theta}_0(t)}{p^2 + \lambda^2 \frac{a^2}{\alpha_1}}, \quad a^2 = \frac{E}{\rho} \qquad (52)$$

coinciding with (33), obtained for small and large values of time, if in (33) $a_*{}^2$ is replaced by a^2 / α_1. Therefore, with such a change, solutions (42), (48) and (49) obtained above are valid for arbitrary values of time.

V. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

Calculations were carried out for the case $\sigma_0(\mathrm{t}) = \sigma_{0c} = const$ (sudden loading) and $\theta_0(\mathrm{t}) = \theta_{0c} = const$ (sudden heating) according to formula (40). On fig. 1 shows graphs of $\frac{\sigma(x,t)}{\sigma_{0c}}$ depending on the dimensionless distance $\bar{x} = \frac{a_*}{a_T{}^2} x$ for four time values $\bar{t} = \frac{a_*{}^2 t}{a_T{}^2} = 0.1, 1, 2.5, 5$ at

$$\beta_1 = \frac{3\alpha\theta_{0c}\rho a_*{}^2}{\sigma_0} \text{*=0.1,0.2, 0.3.}$$

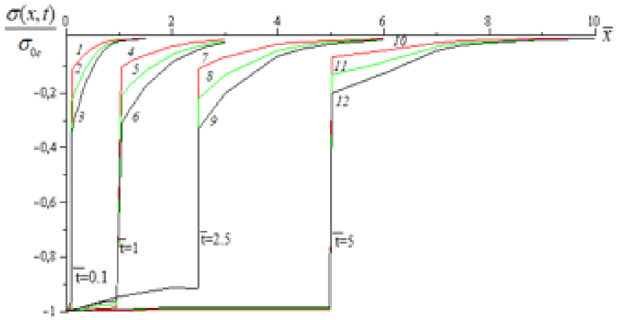

Fig. 1. Curves 1,4,7,10 correspond to $\beta_1 = 0.1$; 2,5,8,11 -- $\beta_1 = 0.2$; 3,6.9,12 -- $\beta_1 = 0.3$.

As an example, Table 1 shows the numerical values of the dimensionless stress corresponding to curve 4 in fig. 1.

Table 1. Numerical values of the dimensionless stress corresponding to curve 4 in fig. 1

x	***0***	***0.1***	***0.5***	***1-0***	***1+0***	***1.5***	***2***	***3***	***4***
$-\frac{\sigma(x,t)}{\sigma_{0c}}$	1	0.9986	0.9934	0.9985	0.0885	0.0467	0.0231	0.0043	0.0041

Let us consider in more detail the effect of sudden heating on the distribution of stress that has arisen during sudden loading of the rod. The function $\sigma_\theta(x,t;\theta_{0c})$ in formula (40), which describes the effect of temperature on the stress distribution, can be written in the following dimensionless form

$$\frac{\sigma_\theta(x,t;\theta_{0c})}{\sigma_{0c}} = \beta_1(y_1 - y_2), \qquad (53)$$

where

$$y_1 = \frac{2}{\pi}\int_0^\infty \frac{(\xi cos(\xi\bar{t}) + \sin(\xi\bar{t}))\sin(\xi\,\bar{\mathrm{x}})}{\xi^2+1}d\xi,$$

$$y_2 = \frac{2}{\pi}\int_0^\infty \frac{\xi\exp(-\xi^2\,\bar{\mathrm{t}})\sin(\xi\,\bar{\mathrm{x}})}{\xi^2+1}d\xi,$$

$$\beta_1 = \frac{3\alpha\theta_{0c}\rho a_*{}^2}{\sigma_{0c}},\quad \bar{t} = \frac{a^2t}{a_T^2},\quad \bar{x} = \frac{ax}{a_T^2}.$$

In (53) y_1 can be reduced to table integrals. Indeed, using the well-known equalities

$$cos(\xi\bar{t})\sin(\xi\,\bar{\mathrm{x}}) = \frac{1}{2}(\sin(\xi(\bar{\mathrm{x}}-\bar{t}) + \sin(\xi(\bar{\mathrm{x}}+\bar{t})),$$

$$\sin(\xi\bar{t})\sin(\xi\,\bar{\mathrm{x}}) = \frac{1}{2}(\cos(\xi(\bar{\mathrm{x}}-\bar{t}) - \cos(\xi(\bar{\mathrm{x}}+\bar{t}))$$

and tabular values of the following integrals [29]

$$\int_0^\infty \frac{x\sin bx}{\omega^2+x^2}dx = \pm\frac{\pi}{2}\exp(-|b\omega|),\ (+\ at\ b>0);$$

$$\int_0^\infty \frac{\cos bx}{x^2+1}dx = \frac{\pi}{2}\exp(-|b|)$$

we get

$$\int_0^\infty \frac{(\xi cos(\xi\bar{t})\sin(\xi\,\bar{x})}{\xi^2+1}d\xi=\begin{cases}\dfrac{\pi}{4}(e^{-|\bar{x}-\bar{t}|}+e^{-|\bar{x}+\bar{t}|}),\ \bar{x}>\bar{t};\\ \dfrac{\pi}{4}(-e^{-|\bar{x}-\bar{t}|}+e^{-|\bar{x}+\bar{t}|}),\ \bar{x}<\bar{t}\end{cases}$$

$$\int_0^\infty \frac{(\sin(\xi\bar{t})\sin(\xi\,\bar{x})}{\xi^2+1}d\xi=\frac{\pi}{4}(e^{-|\bar{x}-\bar{t}|}-e^{-|\bar{x}+\bar{t}|}).$$

It is easy to see that, taking into account the last equalities, y_1 takes the form

$$y_1=e^{-(\bar{x}-\bar{t})}H(\bar{x}-\bar{t})\,. \tag{54}$$

Thus, the effect of sudden heating on stresses on stress is described by the formula

$$\frac{\sigma_\theta(x,t;\theta_{0c})}{\sigma_{0c}}=\beta_1 e^{-|\bar{x}-\bar{t}|}H(\bar{x}-\bar{t})-\\ -\beta_1\frac{2}{\pi}\int_0^\infty\frac{\xi\exp(-\xi^2\,\bar{\mathrm{t}}))\sin(\xi\,\bar{\mathrm{x}})}{\xi^2+1}d\xi, \tag{55}$$

where $H(\bar{x}-\bar{t})$- Heaviside function; $\bar{x}-\bar{t}=0$ - thermoelastic wave front equation. Consider the contribution of two terms $\beta_1 y_1$ and $\beta_2 y_2$ in (53) to the stresses change:

1) $y_1=\exp(-\bar{t}\,(\frac{\bar{x}}{\bar{t}}-1))H(\bar{x}-\bar{t})$ nonzero at any finite distance in the region ahead of the thermoelastic wave front $(\bar{x}>\bar{t})$ and at any finite time value $y_1\big|_{\frac{\bar{x}}{\bar{t}}\to 1+0}\to\beta_1$, but decreases exponentially with increasing $\bar{t}$. The influence of the first term can be noticeable only in a certain region adjacent to the wave front from its front side (Fig. 2).

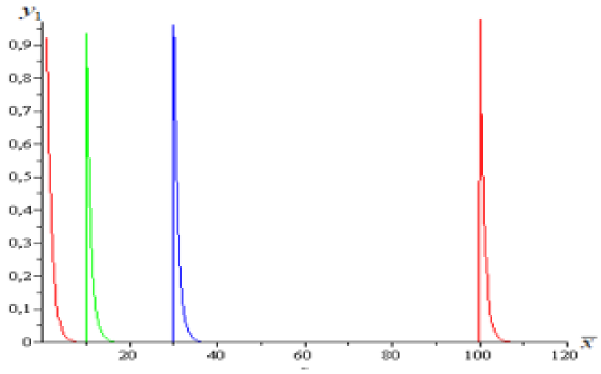

Fig. 2. Curves 1,2,3,4 correspond to $\bar{t}=\dfrac{a^2t}{a_T^2}$=1; 10; 30; 100

2) the contribution of the second term is determined by the integral y_2, which is a function of dimensionless variables $\bar{x}=\dfrac{ax}{a_T^2}$ u $\bar{t}=\dfrac{a^2t}{a_T^2}$. At a fixed value of time $\bar{x}=\dfrac{ax}{a_T^2}$ u $\bar{t}=\dfrac{a^2t}{a_T^2}$, the function $\bar{t}$ first increases with an increase in $\bar{x}$ to a certain maximum (decreases with an increase in $y_2(\bar{t},\bar{x})$), then monotonously decreasing tends to zero at $\bar{x}\to\infty$ (Fig. 3.)

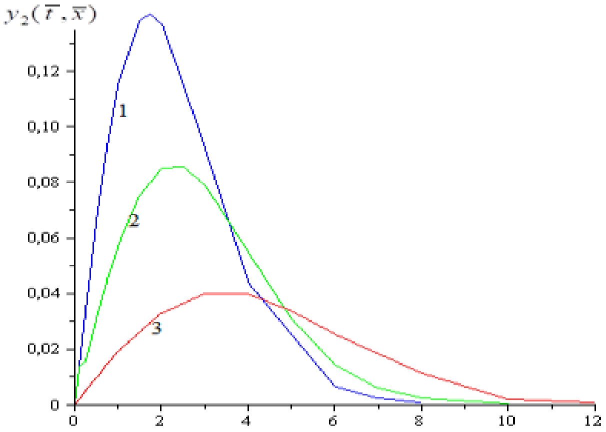

Fig. 3. Curves 1,2,3 correspond to $\bar{t}$ = 1; 2; 5

The contribution $\beta_1 y_2(\bar{t},\bar{x})$ to the stress distribution can be neglected at $\bar{t} \geq 2-2.5$. The validity of this statement follows from the estimate

$$\left|\int_0^\infty \frac{\xi \exp(-\xi^2 \bar{t}) \sin(\xi \bar{x})}{\xi^2+1} d\xi\right| \leq \int_0^\infty \frac{\xi \exp(-\xi^2 \bar{t})}{\xi^2+1} d\xi == \frac{1}{2}\int_0^\infty \frac{\exp(-\xi^2 \bar{t})}{\xi^2+1} d\xi^2.$$

where according to the table of integrals [9]

$$\int_0^\infty \frac{\exp(-\alpha \bar{t})}{\alpha+1} d\alpha = \Gamma(0,\bar{t}) =$$

$$= -E_i(-\bar{t}) = \int_{\bar{t}}^\infty \frac{e^{-\lambda}}{\lambda} d\lambda,$$

and hence $y_2 \leq -\frac{1}{\pi} E_i(-\bar{t})$, where $\Gamma(0,\bar{t})$ is an incomplete gamma function; $-E_i(-\bar{t}) = \int_{\bar{t}}^\infty \frac{e^{-\lambda}}{\lambda} d\lambda$ is an integral exponential function that rapidly decreases with increasing argument:

$$\frac{1}{\pi}(-E_i(-2.5)) = 0.0079$$

calculations to determine the temperature distribution are also made for the special case $\sigma_0(t) = \sigma_{0c} = const$ and $\theta_0(t) = \theta_{0c}$, $= const$ when $\theta(x,t)$ is determined by the formula

$$\theta(x,t) = \theta_{0c} erfc(\frac{x}{2a_T\sqrt{t}}) + \sigma_{0c}\frac{\alpha_1\alpha_2}{E}\int_0^t G_1(x,\tau a_*;t-\tau)d\tau +$$
$$+\frac{\alpha_1\alpha_2}{E}\int_0^t \frac{\partial\sigma_\theta(x,t;\theta_{0c})}{\partial t} G_1(x,\xi;t-\tau)d\xi d\tau, \qquad (56)$$

where

$$\sigma_\theta(x,t;\theta_{0c}) = \beta_1\sigma_{0c}[e^{-|\bar{x}-\bar{t}|}H(\bar{x}-\bar{t}) - \frac{2}{\pi}\int_0^\infty \frac{\xi \exp(-\xi^2 \bar{t})\sin(\xi \bar{x})}{\xi^2+1} d\xi],$$

$$\beta_1 = \frac{3\alpha\theta_{0c}\rho a_*^{\ 2}}{\sigma_{0c}}, \quad \alpha_2 = \frac{3\alpha T_0 K}{\rho c_v + 9\alpha^2 K T_0}$$

The coefficient in front of the first integral in (56) is small, since $\alpha_1\alpha_2\frac{\sigma_{0c}}{E}$ <<1. The coefficient in front of the second integral is a small value of a higher order, since

$$\frac{\alpha_1\alpha_2}{E}\beta_1\sigma_{0c}=\alpha_1\alpha_2\frac{\sigma_{0c}}{E}\frac{3\alpha\theta_{0c}\rho a_*^{\,2}}{\sigma_{0c}}\ll\alpha_1\alpha_2\frac{\sigma_{0c}}{E}$$

Therefore, the third term on the right-hand side of (56) is neglected in calculations as a small value of a higher order. On fig. 4 shows the graphs of the temperature distribution without taking into account the influence of the rod deformation, i.e. with pure thermal conductivity depending on $\overline{x}=\frac{a_1}{a_T^{\,2}}x$ for four times $\overline{t}=\frac{a_*^{\,2}t}{a_T^{\,2}}$:0.1,1,2.5,5.

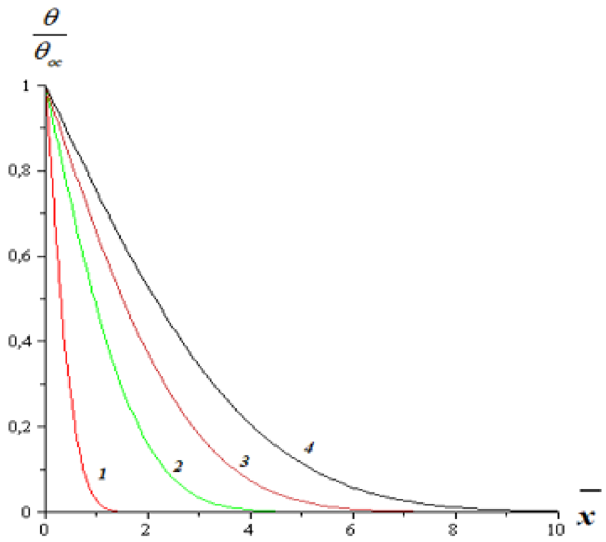

Fig. 4. Curves correspond to time values $\overline{t}=\frac{a_*^{\,2}t}{a_T^{\,2}}$:0.1,1,2.5,5

Figure 5 shows the temperature distribution with and without taking into account the influence of the stress that has arisen during sudden heating.

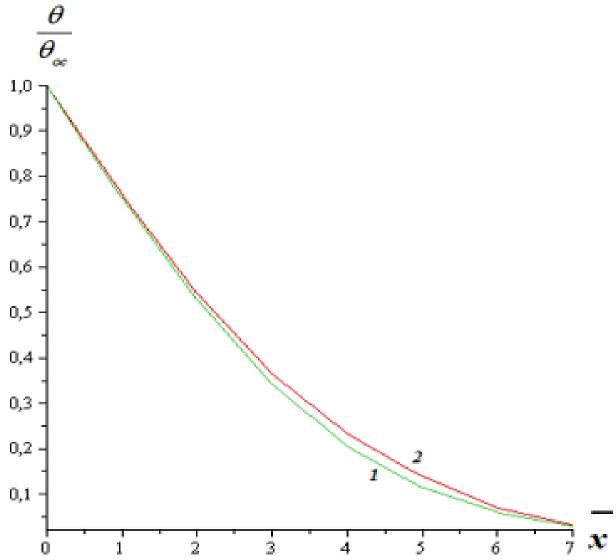

Fig. 5. Function graphs $\theta(\overline{x},5)/\theta_{0c}$: curve 1 - excluding, 2- taking into account the effect of stress on $\overline{t}=a_*^{\,2}t/a_T^{\,2}=5$ for $\sigma_{0c}\alpha_1\alpha_2/E=0.02$ ($\overline{x}=5$ thermoelastic wave front)

As can be seen from the temperature distribution graphs in fig. 4 and fig. 5, with a practically acceptable error, we can assume that there is a temperature propagation front.

VI. CONCLUSION

The following scientific results were obtained in the work:

1. A coupled dynamic problem of thermoelasticity under loading and heating of a semi-infinite rod is formulated, which consists of the equations of motion, the continuity equation, the Dugamel-Neumann relations, and the heat inflow equation. A system of three interrelated equations for stresses, velocities, and temperatures is obtained with the corresponding initial and boundary conditions. In this case, in contrast to existing works, it is not required to introduce a boundary condition with respect to the first derivative of stresses.

2. The boundary value problem is solved by the continuation method followed by the application of the Laplace and Fourier integral transformations.

3. An exact analytical solution of the problem has been obtained:

- for small and large values of time;

- for arbitrary times provided by $3\alpha\alpha_2 << 1$.

4. The influence of temperature on the distribution of stress and velocity, as well as the influence of the latter on the distribution of temperature, has been studied. In this case, we should especially note the case of simultaneous sudden loading and instantaneous heating, in which the expression for the velocity is obtained as the sum of three terms describing the effect, respectively:

- sudden loading;

- temperature changes under the coupled action of sudden loading and instantaneous heating;

- change in temperature due to thermal conductivity during instantaneous heating. In this term, the singularity of the solution of the heat equation manifests itself in the form $v(0,t) = O(t^{-\frac{1}{2}})$.

5. The effect of temperature on the propagation velocity of a thermoelastic wave is revealed. If transverse deformation is taken into account within the framework of one-dimensional motion (stresses, velocities, etc. are considered to be averaged over the cross section of the rod), then the propagation velocity of a thermoelastic wave differs from the propagation velocity of an elastic wave with a certain coefficient (very close in value to unity), which takes into account the influence of the lateral strain and temperature. If we neglect the lateral strains and assume that the parameters of the problem change only along the rod, then similarly, the propagation velocity of a thermoelastic wave differs little from the propagation velocity of a longitudinal wave. Numerical calculations are made, which are presented in the form of graphs. A qualitative and quantitative analysis of the results presented in the figures is given.

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СВЯЗАННАЯ ЗАДАЧА О НАПРЯЖЕНИЯХ ПРИ НАГРУЖЕНИИ ОДНОРОДНОГО ПОЛУБЕСКОНЕЧНОГО ТЕРМОУПРУГОГО СТЕРЖНЯ

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Аннотация. *Обычно полная система уравнений термоупругости состоит из уравнения движения, соотношений Дюгамеля-Неймана, соотношения Коши и уравнения теплового притока. В этом случае обычно краевая задача сводится к системе дифференциальных уравнений для перемещений и температуры с соответствующими начальными и граничными условиями. Связанные задачи термоупругости можно сформулировать как в напряжении, так и в температуре, но для этого требуется второй порядок гладкости тензора напряжений. В связи с этим есть дополнительное условие, касающееся напряжения. В данной работе связанные задачи термоупругости предлагается рассматривать на основе уравнений движения, уравнения неразрывности, соотношения Дюгамеля-Неймана и уравнения теплового притока. При этом была получена система трех взаимосвязанных уравнений для напряжения, скорости и температуры. На основе этой системы рассмотрена задача о напряженно-деформированном состоянии однородного полубесконечного термоупругого стержня под действием динамической термомеханической нагрузки, приложенной к концу стержня. Точное аналитическое решение было получено методом продолжения с последующим применением интегральных преобразований Лапласа и Фурье. Изучено асимптотическое поведение напряжения при малых и больших значениях времени. Изучена также скорость распространения термоупругой волны. Проведены численные расчеты, проанализирован качественный и количественный характер взаимного влияния температуры на распределение напряжений и скоростей сечений стержня.*

Ключевые слова: *напряжение, температура, скорость, стержень, связанная задача, преобразования Фурье и Лапласа.*